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THE APPLICATION OF THE PLK METHOD TO THE INCOMPRESSIBLE LAMINAR AXISYMMETRIC FAR WAKE

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SANTA MONICA • CALIFORNIA

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PREFACE

An attempted asymptotic series solution in terms of inverse powers of the axial distance for the incompressible laminar wake far behind a very slender cylinder breaks down at an early stage. In this Memorandum, the PLK method is applied to the problem and yields a uniformly valid asymptotic solution. The method of approach used should be of interest to researchers in fluid mechanics concerned with the solution of singular perturbation problems, while the results should be of interest to those involved in wake studies.

SUMMARY

The Oseen solution for the wake far behind a very slender cylinder indicates that an asymptotic series solution in inverse powers of the axial distance might be possible. However, if an attempt is made to carry this out, the solution breaks down at an early stage. This occurs when the coefficient of the inverse second power of the axial distance (x^{-2}) exhibits the wrong behavior at the edge of the wake. This difficulty is overcome by applying Lighthill's technique (the PLK method) for rendering approximate solutions uniformly valid. The correct series contains the term $0(\ell n\ x/x^2)$; it was the omission of this term which caused the difficulty encountered in the original series expansion. In addition, it is shown that the next term in the series, $0(x^{-2})$, contains an indeterminate factor. This indeterminacy is due to the occurrence of eigenfunction solutions, but the reason for their existence is not explained. Earlier work on this problem, however, indicates that this is connected with the neglect of the initial velocity profile in obtaining the asymptotic solution. The solution obtained has some relevance to the wake behind any finite axisymmetric body.

CONTENTS

I. INTRODUCTION

The asymptotic solution for the incompressible axisymmetric wake far behind a long thin cylinder (or "needle") was given in Ref. 1. (This reference also contains a discussion of the applicability of the "needle" solution to more general axisymmetric bodies.) The solution was sought in the form of a series in inverse powers of the axial distance. However, the solution so obtained was not a uniformly valid one. This first became evident when the second approximation (corresponding to the negative second power of the axial distance) did not approach zero exponentially at the edge of the wake. A detailed analysis of the problem revealed that the difficulty lay in the assumed form of the expansion and indicated that logarithmic terms had to be included. In particular, it was shown that the term following $O(1/x)$ was $O(\ell n\ x/x^2)$ and not $O(1/x^2)$ as was initially assumed (x is axial distance). The series solution was carried out to the term $O(\ell n\ x/x^2)$, and it was shown that within the context of the asymptotic approach, the term $O(1/x^2)$ could not be completely determined. This indeterminacy was explained as being due to the neglect of one boundary condition in obtaining the asymptotic solution. This last boundary condition is the initial velocity profile at the base of the cylinder. The manifestation of the lost condition in determining the terms in the series was in the form of eigensolutions, that is, complementary solutions of the equations satisfying the boundary conditions on the axis and edge of the wake. Arbitrary multiples of these eigensolutions could be added to certain terms in the expansion, the formulation of the problem being such that there was no way of determining their contribution.

In the present Memorandum, we wish to reconsider this far-wake problem and demonstrate how the correct asymptotic solution can be obtained by using Lighthill's technique for rendering approximations uniformly valid (the PLK method, or "method of strained coordinates"[2]). In addition to correctly predicting the $O(\ell n\ x/x^2)$ term, the technique also determines the explicit form of the eigensolution responsible for the indeterminacy in the $O(1/x^2)$ term.

Application of Lighthill's technique to this problem does yield the solution more readily than the analysis given in Ref. 1; however, its efficiency is diminished by a loss of insight into the basic cause of the indeterminacy appearing in $O(1/x^2)$ and higher-order terms.

II. BASIC EQUATIONS AND ASYMPTOTIC SOLUTION

The equations of motion governing the development of the axisymmetric incompressible wake without pressure gradient can be written as

$$\frac{\partial}{\partial x}(ru) + \frac{\partial}{\partial r}(rv) = 0 \tag{1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) \tag{2}$$

(Although the assumption of constant pressure would seem to limit the resulting solution to very slender bodies, the solution does have some application to the wake far behind any axisymmetric body. See Ref. 1 for a detailed discussion of this point.)

Here, x, r, u, and v are nondimensional variables defined by

$$x = \frac{x_1}{\theta} \qquad r = \left(\frac{U\theta}{\nu}\right)^{\frac{1}{2}}\frac{r_1}{\theta} \qquad u = \frac{u_1}{U} \qquad v = \frac{v_1}{U}$$

where x_1 and r_1 are cylindrical coordinates (x_1 is the axial distance, and r_1 the radial distance measured from the axis of symmetry), u_1 and v_1 are the corresponding velocity components, U is the free-stream velocity, ν is the kinematic viscosity, and θ is the momentum thickness.

Equation (1) can be satisfied by introducing a stream function ψ, defined by

$$u = \frac{1}{r}\frac{\partial\psi}{\partial r} \qquad v = -\frac{1}{r}\frac{\partial\psi}{\partial x} \tag{3}$$

Equations (1) and (2) are to be solved subject to the boundary conditions

$$\psi = 0 \qquad \frac{\partial u}{\partial r} = 0 \qquad \text{at} \qquad r = 0 \tag{4}$$

$$u \to 1 \qquad \text{as} \qquad r \to \infty \tag{5}$$

The solution is most conveniently expressed in terms of a new independent variable ζ, defined by

$$\zeta = \frac{r^2}{4x} \tag{6}$$

which replaces r; we also write the stream function as

$$\psi = xf(x,\zeta) \tag{7}$$

In Ref. 1 the initial approach to an asymptotic solution is made by expanding $f(x,\zeta)$ in the following series:

$$f(x,\zeta) = \sum_{n=0}^{\infty} \frac{f_n(\zeta)}{x^n} \tag{8}$$

However, this method fails when it is found that $f_2(\zeta)$ approaches zero as $\zeta \to \infty$ algebraically and not exponentially, as should be expected on physical grounds. An alternate formulation was used in Ref. 1 to indicate the form of the correct expansion.

In this Memorandum, following the methodology of the PLK technique, we expand the stream function and the x coordinate in the following series:

$$\psi = \beta f(\beta,\zeta) = \beta \sum_{n=0}^{\infty} \frac{f_n(\zeta)}{\beta^n} \tag{9}$$

$$x = \beta + x_1(\beta) + \ldots \tag{10}$$

where $\zeta = r^2/4\beta$.

Substitution of Eq. (9) into Eq. (3) leads to the following expressions for u and v:

$$u = \frac{1}{2} f_\zeta \tag{11}$$

$$v = -\frac{1}{r}[f - \zeta f_\zeta + \beta f_\beta]\frac{d\beta}{dx} \tag{12}$$

From Eq. (10) we find that

$$\frac{d\beta}{dx} = \frac{1}{dx/d\beta} = \frac{1}{1 + x_1'(\beta) + \ldots} \approx 1 - x_1'(\beta) + \ldots \tag{13}$$

assuming that $x_1'(\beta) \ll 1$ when x is large. Then

$$v = -\frac{1}{r}[f - \zeta f_\zeta + \beta f_\beta](1 - x_1'(\beta) + \ldots) \tag{14}$$

Substituting Eqs. (11) and (14) into the momentum equation, Eq. (2), and using Eq. (13), we obtain

$$\zeta f_{\zeta\zeta\zeta} + f_{\zeta\zeta} + \left[\frac{1}{2} ff_{\zeta\zeta} + \frac{1}{2}\beta f_\beta f_{\zeta\zeta} - \frac{1}{2}\beta f_{\beta\zeta} f_\zeta\right](1 - x_1'(\beta) + \ldots) = 0 \tag{15}$$

In terms of the functions $f_n(\zeta)$, the boundary conditions are

$$\left.\begin{array}{llll} f_n(0) = 0 & & & \\ \sqrt{\zeta}\, f_n''(\zeta) \to 0 & \text{as} & \zeta \to 0 & \\ f_o'(\zeta) \to 2 & \text{as} & \zeta \to \infty & \\ f_n'(\zeta) \to 0 & \text{as} & \zeta \to \infty & n \geq 1 \end{array}\right\} \tag{16}$$

If we now substitute the series expansion of $f(\beta,\zeta)$ given in Eq. (9) into Eq. (15) and begin equating coefficients of different powers of β^{-1} to zero, we find that the terms independent of β yield

$$\zeta f_o''' + f_o'' + \frac{1}{2} f_o f_o'' = 0 \tag{17}$$

A solution of this equation satisfying the boundary conditions is $f_o(\zeta) = 2\zeta$, and this term is simply the uniform free stream.

Before any further equations can be obtained, we must choose $x_1'(\beta)$. In Ref. 1 it was found that $f_2'(\zeta)$ goes to zero as ζ^{-2} in the limit of $\zeta \to \infty$, and therefore the expansion was considered to have broken down at this stage. Through an appropriate selection of the value of $x_1'(\beta)$, we shall try to eliminate those terms in $f_2'(\zeta)$ which do not decay exponentially near infinity. Since ψ is expanded in a series in integral powers of β^{-1}, the simplest choice for $x_1'(\beta)$ is

$$x_1'(\beta) = \frac{A}{\beta} \tag{18}$$

where A is a constant.

If the coefficient of β^{-1} is now equated to zero, we obtain (assuming $f_o(\zeta) = 2\zeta$)

$$\zeta f_1''' + (\zeta + 1) f_1'' + f_1' = 0 \tag{19}$$

while the coefficient of β^{-2} yields the equation

$$\zeta f_2''' + (\zeta + 1) f_2'' + 2f_2' + \frac{1}{2} f_1'^2 - \frac{A}{2} (2\zeta f_1'' + 2f_1') = 0 \tag{20}$$

Equation (19) is the same as the equation for $f_1(\zeta)$ found in Ref. 1, and a solution satisfying the boundary conditions is

$$f_1' = C_1 e^{-\zeta} \tag{21}$$

where

$$C_1 = -\frac{U\theta}{2\nu} \tag{22}$$

The momentum thickness θ is a constant and is determined from

$$\pi\theta^2U^2 = 2\pi\int_0^\infty u_1(U - u_1)r_1\, dr_1 \tag{23a}$$

or

$$\theta = \frac{4\nu x}{U}\int_0^\infty u(1 - u)\, d\zeta \tag{23b}$$

It is related to the drag of the body, D, by

$$D = \rho U^2\pi\theta^2 \tag{23c}$$

Substituting Eq. (21) for f_1' into Eq. (20) yields

$$\zeta f_2''' + (\zeta + 1)f_2'' + 2f_2' + \frac{1}{2}C_1^2e^{-2\zeta} + C_1A(\zeta - 1)e^{-\zeta} = 0 \tag{24}$$

The function $f_2(\zeta)$ must satisfy the boundary conditions

$$f_2(0) = 0 \qquad \sqrt{\zeta}\, f_2''(\zeta) \to 0 \quad \text{as} \quad \zeta \to 0$$
$$f_2'(\zeta) \to 0 \quad \text{as} \quad \zeta \to \infty \tag{25}$$

To solve Eq. (24) we set

$$f_2'(\zeta) = (\zeta - 1)e^{-\zeta}\, g_2(\zeta)$$

and $g_2(\zeta)$ then satisfies the equation

$$g_2'' + \frac{2\zeta - (\zeta - 1)^2}{\zeta(\zeta - 1)}\, g_2' = \frac{-C_1^2e^{-\zeta}}{2\zeta(\zeta - 1)} - \frac{C_1A}{\zeta}$$

This is immediately integrable and the solution is

$$g_2(\zeta) = \frac{-C_1^2}{8}\left[\int_\alpha^\zeta \frac{e^{-t}}{t}\,dt + \frac{e^{-\zeta}}{\zeta - 1}\right] + C_2\left[\int_\alpha^\zeta \frac{e^{t}}{t}\,dt - \frac{e^{\zeta}}{\zeta - 1}\right]$$

$$+ C_1 A\left(\ln \zeta - \frac{2}{\zeta - 1}\right)$$

where C_2 and α are arbitrary constants. Hence, $f_2'(\zeta)$ is given by

$$f_2'(\zeta) = \frac{-C_1^2}{8}\left[(\zeta - 1)e^{-\zeta}\int_\alpha^\zeta \frac{e^{-t}}{t}\,dt + e^{-2\zeta}\right] + C_2\left[(\zeta - 1)e^{-\zeta}\int_\alpha^\zeta \frac{e^{t}}{t}\,dt - 1\right]$$

$$+ C_1 A\left[(\zeta - 1)e^{-\zeta}\ln \zeta - 2e^{-\zeta}\right] \tag{26}$$

Since

$$\left.\begin{aligned} \int_\alpha^\zeta \frac{e^{-t}}{t}\,dt &\sim -\frac{e^{-\zeta}}{\zeta} + \text{constant} \\ \int_\alpha^\zeta \frac{e^{t}}{t}\,dt &\sim \frac{e^{\zeta}}{\zeta} + \text{constant} \end{aligned}\right\} \quad \text{as} \quad \zeta \to \infty \tag{27}$$

the only term in Eq. (26) which does not exhibit exponential decay to zero as $\zeta \to \infty$ is the integral multiplied by C_2. Hence, we set $C_2 = 0$. For $\zeta \to 0$

$$\int_\alpha^\zeta \frac{e^{-t}}{t}\,dt \sim \ln \zeta + \text{constant} \tag{28}$$

which means if $f_2'(\zeta)$ is not to become infinite (as $\ln \zeta$) when $\zeta \to 0$, we must choose

$$A = \frac{C_1}{8}$$

With these choices of C_2 and A, Eq. (26) reduces to

$$f_2'(\zeta) = -\frac{C_1^2}{8} e^{-\zeta}\left\{(\zeta - 1)\left[\int_\alpha^\zeta \frac{e^{-t}}{t}\, dt - \ln \zeta\right] + e^{-\zeta} + 2\right\} \quad (29)$$

To check the second of the boundary conditions in Eq. (25), we first calculate

$$f_2''(\zeta) = \frac{C_1^2}{8} e^{-\zeta}\left\{(\zeta - 2)\left[\int_\alpha^\zeta \frac{e^{-t}}{t}\, dt - \ln \zeta\right] + \left(\frac{1}{\zeta} + 1\right) e^{-\zeta} + 3 - \frac{1}{\zeta}\right\}$$

Then

$$\lim_{\zeta\to 0} f_2''(\zeta) = \frac{C_1^2}{8}\left\{-2 \lim_{\zeta\to 0}\left[\int_\alpha^\zeta \frac{e^{-t}}{t}\, dt - \ln \zeta\right] + 4\right\}$$

and, using Eq. (28), we see that

$$\lim_{\zeta\to 0} f_2''(\zeta) = \text{constant}$$

Hence

$$\sqrt{\zeta}\, f_2''(\zeta) \to 0 \qquad \text{as} \qquad \zeta \to 0$$

and the required boundary condition is satisfied.

The last boundary condition is $f_2(0) = 0$. This is satisfied by choosing the constant of integration after integrating $f_2'(\zeta)$. That this can be done follows from the fact that $f_2'(\zeta)$ has been made finite at $\zeta = 0$.

Thus the required solution for $f_2'(\zeta)$ is given by Eq. (29). We note that the lower limit on the integral in the solution is still

arbitrary after satisfying all the boundary conditions. This arbitrariness introduces a term

$$ke^{-\zeta}(\zeta - 1) \tag{30}$$

in $f_2'(\zeta)$ which cannot be determined. The reason is that this function satisfies the required boundary conditions for $f_2'(\zeta)$ both on the axis and on the edge of the wake. If $F'(\zeta)$ represents the function, then the boundary conditions it must satisfy are

$$\sqrt{\zeta}\,F''(\zeta) \to 0 \qquad \text{as} \qquad \zeta \to 0$$

$$F'(\zeta) \to 0 \text{ (exponentially)} \qquad \text{as} \qquad \zeta \to \infty$$

The second of these is seen to be satisfied immediately for $F'(\zeta) = e^{-\zeta}(\zeta - 1)$. For the former, since

$$F''(\zeta) = e^{-\zeta} - (\zeta - 1)e^{-\zeta}$$

we see that

$$F''(0) = 2$$

so that it too is satisfied. (Note that in terms of r and x this eigenfunction can be written

$$e^{-\zeta}(\zeta - 1) = e^{-r^2/4\beta}\left(\frac{r^2}{4\beta} - 1\right)$$

which can be compared to the corresponding eigenfunction for the two-dimensional flat-plate far wake

$$e^{-y^2/4\beta}\left(\frac{y^2}{2\beta} - 1\right)$$

found by Stewartson[3] and Crane.[4])

The solution for the velocity u is now given by

$$u = 1 + \frac{C_1}{2}\frac{e^{-\zeta}}{\beta} - \frac{C_1^2}{16\beta^2} e^{-\zeta}$$

$$\left\{(\zeta - 1)\left[\int_\alpha^\zeta \frac{e^{-t}}{t}\,dt - \ln\zeta\right] + e^{-\zeta} + 2\right\} \tag{31}$$

Integrating Eq. (18) we obtain

$$x_1(\beta) = A \ln\beta + B$$

and substituting into Eq. (10)

$$x(\beta) = \beta + A\ln\beta + B + \dots \tag{32}$$

To invert this, we proceed as follows:

$$\begin{aligned}\beta &= x - A\ln\beta - B + \dots\\ &= x - A\ln(x - A\ln\beta - B + \dots) - B + \dots\\ &= x - A\ln x - B + \dots\end{aligned}$$

Since $A = C_1/8$, we obtain finally

$$\beta \approx x - \frac{C_1}{8}\ln x - B \tag{33}$$

If $\beta(x)$ is introduced into Eq. (31), we obtain

$$u = 1 + \frac{C_1}{2}\frac{e^{-\zeta}}{x} + \frac{C_1^2}{16} e^{-\zeta}\frac{\ln x}{x^2} + \frac{D(\zeta)}{x^2} \tag{34}$$

where $D(\zeta)$ is some function of ζ. This does not yet explicitly display the x dependence of u, since $\zeta = r^2/4\beta$ also involves x. Substituting $\beta(x)$ into ζ we obtain

$$\zeta = \frac{r^2}{4}\left(\frac{1}{x} + \frac{C_1}{8}\frac{\ln x}{x^2} + \frac{B}{x^2} + \ldots\right) \tag{35}$$

from which it can also be shown that

$$e^{-\zeta} = e^{-\bar{\zeta}}\left[1 - \frac{C_1}{8}\bar{\zeta}\frac{\ln x}{x} + 0\left(\frac{1}{x^2}\right)\right]$$

where $\bar{\zeta} = r^2/4x$. Then u can be written

$$u = 1 + \frac{C_1}{2}\frac{e^{-\bar{\zeta}}}{x} - \frac{C_1^2}{16}e^{-\bar{\zeta}}(\bar{\zeta} - 1)\frac{\ln x}{x^2} + 0\left(\frac{1}{x^2}\right) \tag{36}$$

Along the axis, at $\zeta = 0$, this reduces to

$$u_o(x) = 1 + \frac{C_1}{2x} + \frac{C_1^2}{16}\frac{\ln x}{x^2} + 0\left(\frac{1}{x^2}\right) \tag{37}$$

Equations (36) and (37) agree with the corresponding equations given in Ref. 1.

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